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Symmetric vectors and Ricci directions

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Abstract. The relation between symmetric vectors admitted by a Riemannian manifold (a concept which arises in relativistic cosmological models) and the Ricci principal directions is discussed. The existence of two second-order symmetric vectors is found to imply the existence of an infinity of symmetric vectors.

Recently the concept of locally symmetric vector fields in general Riemannian manifolds has been introduced by Walker (1976). He was motivated by his earlier investigations (Walker 1940) on possible laws of orientation of galaxies in the cosmological isotropic model of general relativity. A vector is called symmetric about a point if it is a unit vector which is invariant under all rotations about the vector at the point. A unit vector is defined to have first- (second-) order local symmetry in the manifold if the condition of symmetry is satisfied up to first (second) order about every point of the manifold.

Let M be a Riemannian n-manifold, $n > 3$, with metric tensor g. Then it is shown by Walker that a unit vector field $V \in T(M)$ has first-order local symmetry if it satisfies in local chart $(x^1, \ldots, x^n)$

$$\nabla_\nu V_\mu = \alpha(g_{\mu\nu} - V_\mu V_\nu) \tag{1}$$

for some scalar α (∇_ν denotes covariant derivative with respect to the metric). The vector field has second-order local symmetry if in addition to (1) it satisfies

$$\alpha_\mu = \alpha_\nu V^\nu V_\mu. \tag{2}$$

A unit vector field A is a Ricci principal direction if it satisfies (R denotes the Ricci tensor)

$$R_{\mu\nu} A^\nu = a A_\mu$$

for some scalar a. Now it follows immediately from the integrability conditions of (1) and (2) that every second-order locally symmetric vector field is a Ricci principal direction. Moreover, if a Ricci principal direction has local symmetry, it is necessarily of the second order.

Suppose now that the manifold admits two distinct first-order locally symmetric vector fields $V^{(1)}$ and $V^{(2)}$ (if V is locally symmetric then so is $-V$, and we do not distinguish between them). It can be proved (see Appendix) that the two vector fields $(V^{(1)} \pm V^{(2)})/\sqrt{2}$ are Ricci principal directions. Hence at each point of the manifold the two locally symmetric vectors must lie in the 2-subspace of the tangent space at the

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point spanned by two Ricci principal directions. Several conclusions follow from this result, which restrict the number of first- and second-order locally symmetric vector fields which may be admitted. Note that in the case of a single locally symmetric vector admitted by a manifold, it is not confined to such a 2-space.

Finally we mention a theorem with regard to the generation of locally symmetric vector fields out of given ones. Whereas the existence of two first-order locally symmetric vector fields does not entail the existence of any more such fields, the existence of two second-order locally symmetric fields $V^{(1)}$ and $V^{(2)}$ (with associated scalars $\alpha^{(1)}$ and $\alpha^{(2)}$) can be proved (by direct substitution in the symmetry condition) to imply the existence of the second-order locally symmetric vector field

$$V = p^{(1)}V^{(1)} + p^{(2)}V^{(2)},$$

with

$$p^{(a)} = \psi^{(a)}[(\psi^{(1)})^2 + (\psi^{(2)})^2 + 2\psi^{(1)}\psi^{(2)}\langle V^{(1)}, V^{(2)}\rangle]^{-1/2}, \qquad a = 1, 2,$$

where $\psi^{(a)}$ are two scalars defined by the equations

$$\psi^{(a)}_{\mu} = \alpha^{(a)}\psi^{(a)}V^{(a)}_{\mu}$$

and $\langle\ ,\ \rangle$ denotes the Riemannian inner product.

This, in turn, implies that the manifold admits a two-parameter congruence of second-order locally symmetric vector fields.

In a subsequent paper, the relations between symmetric vectors and Ricci directions will be used in characterisations of manifolds admitting symmetric vectors.

Appendix

We outline the proof of the statement that, if $V^{(1)}$ and $V^{(2)}$ are two first-order locally symmetric vector fields (wth associated scalars $\alpha^{(1)}$ and $\alpha^{(2)}$), then $(V^{(1)} \pm V^{(2)})/\sqrt{2}$ are Ricci principal directions.

Denote

$$\phi = g^{\mu\nu}V^{(1)}_{\mu}V^{(2)}_{\nu}, \qquad \phi^2 < 1.$$

Then equation (1) implies

$$\nabla_{\mu}\phi = (\alpha^{(2)} - \phi\alpha^{(1)})V^{(1)}_{\mu} + (\alpha^{(1)} - \phi\alpha^{(2)})V^{(2)}_{\mu}.$$

Differentiating this expression with respect to x^{ν} and antisymmetrising over μ and ν, we find

$$(1-\phi^2)\nabla_{\mu}\alpha^{(1)} = (\alpha^{(1)}_{\cdot 1} - \phi\alpha^{(1)}_{\cdot 2})V^{(1)}_{\mu} + (\alpha^{(1)}_{\cdot 2} - \phi\alpha^{(1)}_{\cdot 1})V^{(2)}_{\mu}, \qquad \text{(A1)}$$

where

$$\alpha^{(1)}_{\cdot 1} = g^{\mu\nu}V^{(1)}_{\mu}\nabla_{\nu}\alpha^{(1)}, \qquad \alpha^{(1)}_{\cdot 2} = g^{\mu\nu}V^{(2)}_{\mu}\nabla_{\nu}\alpha^{(1)}.$$

A similar expression is obtained for $(1-\phi^2)\nabla_{\mu}\alpha^{(2)}$. The integrability conditions of equation (1) are written in terms of the curvature tensor

$$R_{\mu 1\lambda\tau} = (g_{\mu\lambda} - V^{(1)}_{\mu}V^{(1)}_{\lambda})\nabla_{\tau}\alpha^{(1)} - (g_{\mu\tau} - V^{(1)}_{\mu}V^{(1)}_{\tau})\nabla_{\lambda}\alpha^{(1)} + (\alpha^{(1)})^2(g_{\mu\lambda}V^{(1)}_{\tau} - g_{\mu\tau}V^{(1)}_{\lambda}). \qquad \text{(A2)}$$

Contracting this equation with the metric tensor and employing (A1), we find

$$(1-\phi^2)R_{\mu\nu}V^{(1)\nu} = aV^{(1)}_{\mu} + bV^{(2)}_{\mu}, \tag{A3}$$

with

$$a = (1-\phi^2)[(n-1)(\alpha^{(1)})^2 + \alpha^{(1)}_{\cdot 1}] + (n-2)(\alpha^{(1)}_{\cdot 1} - \phi\alpha^{(1)}_{\cdot 2}),$$

$$b = (n-2)(\alpha^{(1)}_{\cdot 2} - \phi\alpha^{(1)}_{\cdot 1}).$$

Transvection of equation (A2) with $V^{(2)\mu}V^{(2)\lambda}V^{(1)\tau}$ and antisymmetrisation over the subscripts 1 and 2 yield

$$(\alpha^{(1)})^2 + \alpha^{(1)}_{\cdot 1} = (\alpha^{(2)})^2 + \alpha^{(2)}_{\cdot 2}, \tag{A4}$$

whereas a similar procedure in (A3) gives (in view of (A4))

$$\alpha^{(2)}_{\cdot 1} - \alpha^{(1)}_{\cdot 2} = \phi(\alpha^{(2)}_{\cdot 2} - \alpha^{(1)}_{\cdot 1}). \tag{A5}$$

Direct substitution of equations (A4) and (A5) in (A3) and its counterpart (replacing $V^{(1)} \leftrightarrow V^{(2)}$), leads to the relation

$$(1-\phi^2)R_{\mu\nu}(V^{(1)\nu} \pm V^{(2)\nu}) = (a \pm b)(V^{(1)}_{\mu} \pm V^{(2)}_{\mu}),$$

which completes the proof.

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